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Construction of rectangular block designs applicable in two-factorial experiments

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SUMMARY

This paper deals with planning of experiments in partially balanced incomplete block designs with rectangular association scheme (RT PBIB). Some methods for construction of RT PBIB which can be applied to two-factor experiments are given. Incidence matrices of these designs can be obtained from Kronecker product generalization of the selected matrices. Attention is called to statistical properties of these designs connected with the estimation of the natural and elementary treatment contrasts. Catalogue of RT PBIB designs for 6, 8, 10 and 12 treatments whose number of treatment replications r satisfies inequality $2 \leq r \leq 15$ is included.

KEY WORDS: two-factorial experiments, Kronecker product of matrices, balanced matrices, RT PBIB designs

1. Introduction

The most important block designs include partially balanced incomplete block designs based on the rectangular association scheme with three association classes (RT PBIB). The RT PBIB design is such an experimental plan in which v treatments are arranged in b blocks of size k, while each treatment occurs r times and in the given block it can occur at most once. Furthermore, the treatments may be divided into m groups of v^* distinct treatments each ($v = mv^*$) such that two treatments belonging to the same group occur in λ_1 blocks and each pair of treatments from different groups with equal numbers in groups occurs in λ_2 blocks, otherwise they occur in λ_3 blocks. The numbers $v = mv^*$, b, r, k, $n_1 = v^* - 1$, $n_2 = m - 1$, $n_3 = (v^* - 1)(m - 1)$, λ_1, λ_2 and λ_3 are called the parameters of the design.

If $\mathbf{N} = (n_{ij})$ is a $(v \times b)$ incidence matrix of RT PBIB design, then from the above definition we have

$$\mathbf{NN}' = r\mathbf{A}_0 + \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2 + \lambda_3\mathbf{A}_3, \tag{1.1}$$

where $\mathbf{A}_0 = \mathbf{I}_v$, $\mathbf{A}_1 = \mathbf{I}_m \otimes (\mathbf{J}_{v*} - \mathbf{I}_{v*})$, $\mathbf{A}_2 = (\mathbf{J}_m - \mathbf{I}_m) \otimes \mathbf{I}_{v*}$, $\mathbf{A}_3 = (\mathbf{J}_m - \mathbf{I}_m) \otimes (\mathbf{J}_{v*} - \mathbf{I}_{v*})$, while $\mathbf{I}_x$ is the unit matrix of order x, $\mathbf{J}_x$ is a $(x \times x)$-matrix of ones and $\otimes$ denotes the Kronecker product of matrices. Note that $\mathbf{A}_1\mathbf{1}_v = n_1\mathbf{1}_v$, $\mathbf{A}_2\mathbf{1}_v = n_2\mathbf{1}_v$, $\mathbf{A}_3\mathbf{1}_v = n_3\mathbf{1}_v$, where $\mathbf{1}_v$ denotes vector of v ones.

The RT PBIB designs can be recommended for experimenters due to two basic reasons. They possess favourable statistical properties, and furthermore, they can be used in two-factorial experiments. These problems, connected with the estimability of treatment contrasts, are explained in the third part of this paper.

Construction of RT PBIB designs consists in finding incidence matrix $\mathbf{N}$ satisfying (1.1). In Section 4, plans of these designs for $2 \leq r \leq 15, 2 \leq k \leq 11$ and $v \leq 12$ are given. These designs can be obtained by a method presented in the next section of this paper. This method is based on the notion of extending the binary case of Kronecker product of matrices (Brzeskwiniewicz et al., 1996).

2. Constructions of RT PBIB designs

We are going to use three notions: balanced block designs, balanced matrices and generalization of the Kronecker product of matrices.

DEFINITION 2.1. (see, e.g., Raghavarao, 1971). A balanced incomplete block design is an arrangement of v^* treatments in b^* blocks of sizes k^* such that every treatment occurs r^* times and every pair of distinct treatments is contained in exactly λ^* blocks.

Ceranka and Goszczurna (1994) give a complete list of these incidence matrices for $v^* < 20$, $r^* \leq 15$, $2 \leq k^* \leq v^*/2$ and an additional remark about the construction with $v^*/2 < k^* < v^* - 1$.

The following definition is due to Shah (1959):

DEFINITION 2.1. Let $\mathbf{A}$ be a $(m \times n)$ matrix whose elements take the s values $1, 2, ..., s$. We shall call matrix $\mathbf{A}$ a balanced matrix in s integers if the following conditions are satisfied:

1^o. The number of times the integer $p (p = 1, 2, ..., s)$ occurs in a column is the same for all the columns and it is equal α_p.

2^o. The number of times the integer $p (p = 1, 2, ..., s)$ occurs in a row is the same for all the rows and it is equal to β_p .

3^o. The number of times the combination p and q (or q and p, $p \neq q$, p, $q =$

$1, 2, ..., s$) occurs in a pair of rows is the same for all the pairs of rows and it is equal to γ_{pq}.

Shah (1959) gives a complete list of these matrices for m, $n \leq 15$ and $s \geq 3$. When we take a smaller number s, according to lemma 2.3 (quoted in the work mentioned above), it is not difficult. Balanced matrix $\mathbf{A}$ will be sometimes written in the form $\mathbf{A}(1, 2, ..., s)$, underlining that there occur all elements $1, 2, ..., s$. Matrices $\mathbf{A}(1, 2, 3, 4, ..., 4)$, $\mathbf{A}(1, 2, 3, ..., 3)$ and $\mathbf{A}(1, 2, ..., 2)$ are thus balanced matrices in $s = 4$, $s = 3$ and $s = 2$ integer, respectively. Also $\mathbf{A}(1, 2) = \mathbf{N}_{v,b} + 2(\mathbf{J}_{v,b} - \mathbf{N})$, where $\mathbf{N}$ is the incidence matrix of a BIB design, is a balanced matrix in $s = 2$ integers. Only these matrices will be used in this paper.

The following definition was used by Brzeskwiniewicz et. al. (1996) (after Shah, 1959):

DEFINITION 2.3. Let $\mathbf{A} = \mathbf{A}(1, 2, ..., s)$ be a $(m \times n)$ balanced matrix in s integers and let there be s $(v^* \times b^*)$ matrices $\mathbf{N}_i$ $(i = 1, ..., s)$. If we replace the integer i in the matrix $\mathbf{A}$ by the matrix $\mathbf{N}_i$, then we shall call it (i.e., matrix $\mathbf{A}(\mathbf{N}_1, \mathbf{N}_2, ..., \mathbf{N}_s)$) an extended binary Kronecker product (EBKP) of matrices $\mathbf{A}$ and $\mathbf{N}_i$ and we denote it by $\mathbf{A}\bar{\otimes}(\mathbf{N}_1, \mathbf{N}_2, ..., \mathbf{N}_s)$.

The following theorems will be used in this paper:

THEOREM 2.1. *Let* $\mathbf{A} = \mathbf{A}(1, 2, 3, 4)$, $\mathbf{N}_{v^*,b^*}$, $\mathbf{0}_{v^*,b^*}$, $\mathbf{J}_{v^*,b^*}$ *be, respectively, a* $(m \times n)$ *balanced matrix in* $s = 4$ *integers, a* $(v^* \times b^*)$ *incidence matrix of BIB design with parameters* $v^*, b^*, r^*, k^*, \lambda^* \neq 0$, *a* $(v^* \times b^*)$ *null matrix and a* $(v^* \times b^*)$ *matrix of ones. Then*

$$\mathbf{N}^{(1)} = \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{0}_{v^*,b^*}, \mathbf{J}_{v^*,b^*}, \mathbf{J}_{v^*,b^*} - \mathbf{N}_{v^*,b^*})$$

is an incidence matrix of RT PBIB design with parameters:
$v = mv^*$, $b = nb^*$, $r = \beta_1 r^* + \beta_3 b^* + \beta_4(b^* - r^*)$, $k = \alpha_1 k^* + \alpha_3 v^* + \alpha_4(v^* - k^*)$,
$n_1 = v^* - 1$, $n_2 = m - 1$, $n_3 = (v^* - 1)(m - 1)$, $\lambda_1 = \beta_1\lambda^* + \beta_3 b^* + \beta_4(b^* - 2r^* + \lambda^*)$,
$\lambda_2 = \gamma_{11}r^* + \gamma_{13}r^* + \gamma_{34}(b^* - r^*) + \gamma_{33}b^* + \gamma_{44}(b^* - r^*)$,
$\lambda_3 = \gamma_{11}\lambda^* + \gamma_{13}r^* + \gamma_{14}(r^* - \lambda^*) + \gamma_{34}(b^* - r^*) + \gamma_{33}b^* + \gamma_{44}(b^* - 2r^* + \lambda^*)$,
where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, $\beta_1, \beta_2, \beta_3, \beta_4$, $\gamma_{11}, \gamma_{13}, \gamma_{14}, \gamma_{33}, \gamma_{34}, \gamma_{44}$ *are the same as in Definition 2.2.*

Proof. Follows immediately after substituting $\mathbf{N}^{(1)}$ in place of $\mathbf{N}$ in (1.1). □

We can also prove that Theorem 2.1 is true for $s = 3$ and $s = 2$, which yields the following result.

COROLLARY 2.1. *If* $\mathbf{A} = \mathbf{A}(1, 2, 3)$ *is a balanced matrix in* $s = 3$ *integers, then*

$$\mathbf{N}^{(1)}_{1,2,3} = \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{0}_{v^*,b^*}, \mathbf{J}_{v^*,b^*}),$$

$$\begin{aligned}\mathbf{N}^{(1)}_{1,2,4} &= \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{0}_{v^*,b^*}, \mathbf{J}_{v^*,b^*} - \mathbf{N}_{v^*,b^*}),\ \textit{if}\ \gamma_{11} \neq 0,\\ \mathbf{N}^{(1)}_{1,3,4} &= \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{J}_{v^*,b^*}, \mathbf{J}_{v^*,b^*} - \mathbf{N}_{v^*,b^*})\end{aligned}$$

are incidence matrices of RT PBIB designs with parameters as in Theorem 1.1 with the omission of components not containing indices mentioned in $\mathbf{N}^{(1)}$.

Condition $\gamma_{11} \neq 0$ in Corollary 2.1 and omission of the remaining matrices $\mathbf{N}^{(1)}_{1,2,3}$, $\mathbf{N}^{(1)}_{1,2,4}$, and $\mathbf{N}^{(1)}_{1,3,4}$ follow from the exclusion of the particular case $\lambda_2 = \lambda_3$.

COROLLARY 2.2. *If* $\mathbf{A} = \mathbf{A}(1,2)$ *is a balanced matrix in* $s = 2$ *integers, then*

$$\begin{aligned}\mathbf{N}^{(1)}_{1,2} &= \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{0}_{v^*,b^*}),\\ \mathbf{N}^{(1)}_{1,3} &= \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{J}_{v^*,b^*}),\ \textit{if}\ \gamma_{11} \neq 0,\\ \mathbf{N}^{(1)}_{1,4} &= \mathbf{A}\bar{\otimes}(\mathbf{N}_{v^*,b^*}, \mathbf{J}_{v^*,b^*} - \mathbf{N}_{v^*,b^*})\end{aligned}$$

are incidence matrices of RT PBIB designs with parameters obtained as in Corollary 2.1.

THEOREM 2.2. *If* $\mathbf{A} = \mathbf{A}(1,2,3,4)$ *is a* $(m \times n)$ *balanced matrix in* $s = 4$ *integers with* $\gamma_{11} \neq 0$ *or* $\gamma_{44} \neq 0$ *then*

$$\mathbf{N}^{(2)} = \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{0}_{v^*}, \mathbf{J}_{v^*}, \mathbf{J}_{v^*} - \mathbf{I}_{v^*})$$

is an incidence matrix of RT PBIB design with parameters
$v = mv^*$, $b = nb^*$, $r = \beta_1 + \beta_3 v^* + \beta_4(v^* - 1)$, $k = \alpha_1 + \alpha_3 v^* + \alpha_4(v^* - 1)$,
$n_1 = v^* - 1$, $n_2 = m - 1$, $n_3 = (v^* - 1)(m - 1)$, $\lambda_1 = \beta_3 v^* + \beta_4(v^* - 1)$,
$\lambda_2 = \gamma_{11} + \gamma_{33} v^* + \gamma_{44}(v^* - 1) + \gamma_{13} + \gamma_{34}(v^* - 1)$,
$\lambda_3 = \gamma_{33} + \gamma_{44}(v^* - 2) + \gamma_{13} + \gamma_{14} + \gamma_{34}(v^* - 1)$,
where $\alpha_1, \alpha_3, \alpha_4$, $\beta_1, \beta_3, \beta_4$, $\gamma_{11}, \gamma_{13}, \gamma_{14}, \gamma_{33}, \gamma_{34}, \gamma_{44}$ *are the same as in Definition 2.2.*

Proof. Follows immediately after substituting $\mathbf{N}^{(2)}$ in place of $\mathbf{N}$ in (1.1). □

When $s = 3$ and $s = 2$, Theorem 2.1 yields the following

COROLLARY 2.3. *If* $\mathbf{A} = \mathbf{A}(1,2,3)$ *is a balanced matrix in* $s = 3$ *integers, then*

$$\begin{aligned}\mathbf{N}^{(2)}_{1,2,3} &= \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{0}_{v^*,b^*}, \mathbf{J}_{v^*}),\ \textit{if}\ \gamma_{11} \neq 0\\ \mathbf{N}^{(2)}_{1,2,4} &= \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{0}_{v^*,b^*}, \mathbf{J}_{v^*} - \mathbf{I}_{v^*}),\ \textit{if}\ \gamma_{11} \neq 0\ \textit{or}\ \gamma_{33} \neq 0,\\ \mathbf{N}^{(2)}_{1,3,4} &= \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{J}_{v^*}, \mathbf{J}_{v^*} - \mathbf{I}_{v^*}),\ \textit{if}\ \gamma_{11} \neq 0\ \textit{or}\ \gamma_{33} \neq 0\end{aligned}$$

are incidence matrices of RT PBIB designs.

COROLLARY 2.4. *If* $\mathbf{A} = \mathbf{A}(1,2)$ *is a balanced matrix in* $s = 2$ *integers, then*

$$\begin{aligned} \mathbf{N}_{1,3}^{(2)} &= \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{J}_{v^*}), \text{ if } \gamma_{11} \neq 0, \\ \mathbf{N}_{1,4}^{(1)} &= \mathbf{A}\bar{\otimes}(\mathbf{I}_{v^*}, \mathbf{J}_{v^*} - \mathbf{I}_{v^*}), \text{ if } \gamma_{11} \neq 0 \text{ or } \gamma_{22} \neq 0, \end{aligned}$$

are incidence matrices of RT PBIB designs. Parameters of the designs obtained in Corollary 2.3 and 2.4 are analogical as in Corollary 2.1.

3. Estimability of treatment contrasts in RT PBIB designs

RT PBIB designs have favourable statistical properties connected with the estimability of the basic and elementary treatment contrasts. Note first that the association matrix (1.1) can be equivalently written in the spectral form

$$\mathbf{NN}' = \rho_0\mathbf{X}_0 + \rho_1\mathbf{X}_1 + \rho_2\mathbf{X}_2 + \rho_3\mathbf{X}_3, \tag{3.1}$$

where $\rho_0 = r + (v^* - 1)\lambda_1 + (m-1)\lambda_2 + (v^* - 1)(m-1)\lambda_3 = rk$, $\rho_1 = r - \lambda_1 + (m - 1)\lambda_2 - (m-1)\lambda_3$, $\rho_2 = r + (v^* - 1)\lambda_1 - \lambda_2 - (v^* - 1)\lambda_3$ and $\rho_3 = r - \lambda_1 - \lambda_2 - \lambda_3$ are latent roots of $\mathbf{NN}'$ with multiplicities equal to 1, $v^* - 1, m - 1$ and $(v^* - 1)(m - 1)$, respectively, and

$$\begin{aligned} \mathbf{X}_0 &= \frac{1}{v}(\mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3) = \mathbf{J}_v/v, \\ \mathbf{X}_1 &= \frac{1}{v}((v^* - 1)\mathbf{A}_0 + \mathbf{A}_1 + (v^* - 1)\mathbf{A}_2 - \mathbf{A}_3) = \mathbf{J}_m/m \otimes (\mathbf{I}_{v^*} - \mathbf{J}_{v^*}/v^*), \\ \mathbf{X}_2 &= \frac{1}{v}((m-1)\mathbf{A}_0 + (m-1)\mathbf{A}_1 - \mathbf{A}_2 - \mathbf{A}_3) = (\mathbf{I}_m - \mathbf{J}_m/m) \otimes \mathbf{J}_{v^*}/v^*, \\ \mathbf{X}_3 &= \frac{1}{v}((m-1)(v^* - 1)\mathbf{A}_0 - (m-1)\mathbf{A}_1 - (v^* - 1)\mathbf{A}_2 - \mathbf{A}_3) = \\ &= (\mathbf{I}_m - \mathbf{J}_m/m) \otimes (\mathbf{I}_{v^*} - \mathbf{J}_{v^*}/v^*). \end{aligned}$$

In the intra-block analysis of variance, a great role is played by matrix $\mathbf{C}$, which in the case of $\mathbf{N1}_b = r\mathbf{1}_v$ and $\mathbf{N}'\mathbf{1}_v = k\mathbf{1}_b$ (RT PBIB designs satisfy this) takes the form $\mathbf{C} = r\mathbf{I}_v - \mathbf{NN}'/k$. Hence, from (3.1) we have

$$\mathbf{C} = \mu_0\mathbf{X}_0 + \mu_1\mathbf{X}_1 + \mu_2\mathbf{X}_2 + \mu_3\mathbf{X}_3, \tag{3.2}$$

where $\mu_i = r - \rho_i/k$, $i = 0, 1, 2, 3$, are latent roots of $\mathbf{C}$ with multiplicities as in (3.1).

From (1.1), (3.1) and (3.2) it follows that the Moore-Penrose inverse of $\mathbf{C}$ can be expressed as

$$\mathbf{C}^{+} = \sum_{i\in I} \mu_i^{-1}\mathbf{X}_i \qquad \text{or} \qquad \mathbf{C}^{+} = \sum_{i=0}^{3} d_i\mathbf{A}_i \tag{3.3}$$

where $I = \{j : \mu_j \neq 0\}$ and $d_i = \sum_{j\in J}\mu_j^{-1}z^{ji}$, while $z^{10} = \frac{v^*-1}{v}$, $z^{20} = \frac{m-1}{v}$, $z^{30} = \frac{(v^*-1)(m-1)}{v}$, $z^{11} = -\frac{1}{v}$, $z^{21} = \frac{m-1}{v}$, $z^{31} = -\frac{m-1}{v}$, $z^{12} = \frac{v^*-1}{v}$, $z^{22} = -\frac{1}{v}$, $z^{32} = \frac{1-v^*}{v}$, $z^{13} = -\frac{1}{v}$, $z^{23} = -\frac{1}{v}$, $z^{33} = \frac{1}{v}$.

In the case of connected designs we have $I = \{1,2,3\}$ and therefore $d_0 = \frac{1}{v}(\frac{v^*-1}{\mu_1} + \frac{m-1}{\mu_2} + \frac{(m-1)(v^*-1)}{\mu_3})$, $d_1 = \frac{1}{v}(-\frac{1}{\mu_1} + \frac{m-1}{\mu_2} - \frac{m-1}{\mu_3})$, $d_2 = \frac{1}{v}(\frac{v^*-1}{\mu_1} - \frac{1}{\mu_2} - \frac{v^*-1}{\mu_3})$, $d_3 = \frac{1}{v}(-\frac{1}{\mu_1} - \frac{1}{\mu_2} + \frac{1}{\mu_3})$.

Consider the vectors:

$$\begin{aligned}
\mathbf{p}_{1i} &= \mathbf{p}_i^m \otimes \mathbf{1}_{v^*} / \parallel \mathbf{p}_i^m \otimes \mathbf{1}_{v^*} \parallel,\ i = 1,...,m-1,\\
\mathbf{p}_{2j} &= \mathbf{1}_m \otimes \mathbf{p}_j^{v^*} / \parallel \mathbf{1}_m \otimes \mathbf{p}_j^{v^*} \parallel,\ j = 1,...,v^*-1,\\
\mathbf{p}_{3ij} &= \mathbf{p}_i^m \otimes \mathbf{p}_j^{v^*} / \parallel \mathbf{p}_i^m \otimes \mathbf{p}_j^{v^*} \parallel,
\end{aligned}$$

where $\mathbf{p}_i^m$ are any collections of $m-1$ mutually orthogonal vectors with m components, satisfying the condition $(\mathbf{p}_i^m)'\mathbf{1}_m = 0$. Similarly, the vectors $\mathbf{p}_j^{v^*}$ are any collections of v^*-1 vectors with v^* components satisfying $(\mathbf{p}_j^{v^*})'\mathbf{1}_{v^*} = 0$, and $\parallel \mathbf{x} \parallel = \sqrt{\mathbf{x}'\mathbf{x}}$. For example we can assume:

$$\mathbf{p}_1^m = [m-1,-1,...,-1]', \mathbf{p}_2^m = [0, m-2, -1, ..., -1]', ..., \mathbf{p}_{m-1}^m = [0,...,0,1,-1]'$$

and

$$\mathbf{p}_1^{v^*} = [v^*-1,-1,...,-1]', \mathbf{p}_2^{v^*} = [0, v^*-2,-1,...,-1]', ..., \mathbf{p}_{v^*-1}^{v^*} = [0,...,0,1,-1]'.$$

The following equalities hold:

$$\mathbf{C}\mathbf{p}_{1i} = \mu_2\mathbf{p}_{1i},\ \mathbf{C}\mathbf{p}_{2j} = \mu_1\mathbf{p}_{2j},\ \mathbf{C}\mathbf{p}_{3ij} = \mu_3\mathbf{p}_{3ij}, \tag{3.4}$$

i.e., $\mathbf{p}_{1i}$, $\mathbf{p}_{2i}$ and $\mathbf{p}_{3ij}$ are latent vectors of $\mathbf{C}$.

It is known that in fixed linear models for block design variances of the estimators of treatment contrasts $\mathbf{p}_{1i}'\mathbf{t}$, $\mathbf{p}_{2j}'\mathbf{t}$ and $\mathbf{p}_{3ij}'\mathbf{t}$,when $\mu_i \neq 0$, are equal (cf. 3.4) to

$$\mathrm{Var}(\widehat{\mathbf{p}_{1i}'\mathbf{t}}) = \frac{\sigma^2}{\mu_2},\quad \mathrm{Var}(\widehat{\mathbf{p}_{2j}'\mathbf{t}}) = \frac{\sigma^2}{\mu_1},\quad \mathrm{Var}(\widehat{\mathbf{p}_{3ij}'\mathbf{t}}) = \frac{\sigma^2}{\mu_3}, \tag{3.5}$$

where $\mathbf{t}$ is an unknown column vector of treatment parameters and σ^2 denotes the error variance of the intra-block analysis. From (3.5) it follows that contrasts from the first group (determined by vectors $\mathbf{p}_{1i}$, $i = 1,...,m-1$) are estimable with the same variances (or all are nonestimable when $\mu_2 = 0$), a similar situation is with contrasts from the second group (determined by vectors $\mathbf{p}_{2j}$, $j = 1, ..., v^* - 1$) and from the third group (determined by vectors $\mathbf{p}_{3ij}$, $i = 1, ..., m - 1$, $j = 1, ..., v^* - 1$). These contrasts will be called the basic contrasts. The variances $\frac{\sigma^2}{\mu_i}$ are smaller (what is a favourable property) if and only if μ_i are greater. The above situation can be applied to select an incidence matrix $\mathbf{N}$. This is interesting in case of two-factorial experiments, in which m denotes the number of levels for factor $\mathbf{A}$, and v^* denotes the number of levels for factor $\mathbf{B}$. Then, we assume as treatments the combinations of the levels of the factors and we assume the lexicographical order $\mathbf{A}_1\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_2, ..., \mathbf{A}_m\mathbf{B}_{v^*}$. Then, vectors $\mathbf{p}_{1i}$ determine the main effects of the factor $\mathbf{A}$, vectors $\mathbf{p}_{2j}$ determine the main effects of the factor $\mathbf{B}$, and vectors $\mathbf{p}_{3ij}$ determine the interaction between the factors $\mathbf{A}$ and $\mathbf{B}$ ($\mathbf{A} \times \mathbf{B}$).

Let us consider the elementary contrasts, i.e. contrasts determined from vectors with only two nonzero components: 1 and -1. Variances of estimators of these contrasts are equal to: $2\sigma^2(d_0 - d_1)$, $2\sigma^2(d_0 - d_2)$ or $2\sigma^2(d_0 - d_3)$, depending on whether the contrast refers to a pair of treatments occurring in λ_1, λ_2 or λ_3 blocks. Variances for any pair occurring in the same number of blocks are equal. In the selection of incidence matrix $\mathbf{N}$, we should pay attention to the fact that the group of elementary contrasts being in the focus of the investigator s interest, has a smaller value of $d_0 - d_1$ ($i = 1, 2, 3$). In case of two-factor experiments, the contrasts from the first group (for λ_1) serve for the comparison of any two levels of factor $\mathbf{B}$ (for a fixed level of factor $\mathbf{A}$). Similarly, the contrasts of the second group (for λ_2) can compare any two levels of factor $\mathbf{A}$ (for a fixed level of factor $\mathbf{B}$); and for the third group (for λ_3), the contrasts can compare two selected combinations $\mathbf{A}_i\mathbf{B}_j$ and $\mathbf{A}_{i'}\mathbf{B}_{j'}$ (with $i \neq i'$ and $j \neq j'$).

Summing up, one can state that favourable statistical properties of the discussed block designs consist in the fact that basic and elementary contrasts can be divided into three groups. All contrasts belonging to one group are either estimable with the same variances or are nonestimable. In two-factorial experiments, an additional convenience consists in the fact that the particular groups of contrasts, both the basic and the elementary ones, are connected with factor $\mathbf{A}$, factor $\mathbf{B}$, and the $\mathbf{A} \times \mathbf{B}$ interaction, respectively. This permits to solve in a simple way the problem of experiment planning: we select a design with the highest μ_i value (for basic contrasts) or the smallest $d_0 - d_i$ value (for elementary contrasts). The μ_i and $d_0 - d_i$ values are calculated as in (3.2) and (3.3) on the basis of design parameters given in the plans in the next chapter.

4. Parameters and plans of RT PBIB designs

Below we present parameters and plans of RT PBIB designs for 6, 8, 10 and 12 treatments. This is a full list of designs that can obtained by methods presented in this paper for $v \leq 12$ and $2 \leq r, k \leq 15$. No designs have been obtained for $v = 9$, and we omit situations: $\lambda_1 = \lambda_2$ and $m = v^*$, $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_3$. Designs satisfying the above conditions can be reduced to group divisible designs whose plans were given by Clatworthy (1973).

$v = 6,\ m = 3,\ v^* = 2,\ b = 6,\ r = 2,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 1$
(1,6),(2,5),(4,5),(3,6),(2,3),(1,4)

$v = 6,\ m = 3,\ v^* = 2,\ b = 12,\ r = 4,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 2$
(1,6),(2,5),(4,5),(3,6),(2,3),(1,4)
This plan should be repeated twice

$v = 6,\ m = 3,\ v^* = 2,\ b = 18,\ r = 6,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 3$
(1,6),(2,5),(4,5),(3,6),(2,3),(1,4)
This plan should be repeated three times

$v = 6,\ m = 3,\ v^* = 2,\ b = 24,\ r = 8,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 4$
(1,6),(2,5),(4,5),(3,6),(2,3),(1,4)
This plan should be repeated four times

$v = 6,\ m = 3,\ v^* = 2,\ b = 30,\ r = 10,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 5$
(1,6), (2,5), (4,5), (3,6), (2,3), (1,4)
This plan should be repeated five times

$v = 6,\ m = 3,\ v^* = 2,\ b = 36,\ r = 12,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 6$
(1,6), (2,5), (4,5), (3,6), (2,3), (1,4)
This plan should be repeated six times

$v = 6,\ m = 3,\ v^* = 2,\ b = 42,\ r = 14,\ k = 2,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 7$
(1,6), (2,5), (4,5), (3,6), (2,3), (1,4)
This plan should be repeated seven times

$v = 6,\ m = 2,\ v^* = 3,\ b = 6,\ r = 3,\ k = 3,\ n_1 = 2,\ n_2 = 1,\ n_3 = 2,\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2$
(1,2,6), (1,3,5), (2,3,4), (3,4,5), (2,4,6), (1,5,6)

$v = 6,\ m = 2,\ v^* = 3,\ b = 12,\ r = 6,\ k = 3,\ n_1 = 2,\ n_2 = 1,\ n_3 = 2,\ \lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = 4$
(1,2,6), (1,3,5), (2,3,4), (3,4,5), (2,4,6), (1,5,6)
This plan should be repeated twice

$v = 6,\ m = 2,\ v^* = 3,\ b = 18,\ r = 9,\ k = 3,\ n_1 = 2,\ n_2 = 1,\ n_3 = 2,\ \lambda_1 = 3,\ \lambda_2 = 0,\ \lambda_3 = 6$
(1,2,6), (1,3,5), (2,3,4), (3,4,5), (2,4,6), (1,5,6)
This plan should be repeated three times

$v = 6,\ m = 2,\ v^* = 3,\ b = 24,\ r = 12,\ k = 3,\ n_1 = 2,\ n_2 = 1,\ n_3 = 2,\ \lambda_1 = 4,\ \lambda_2 = 0,\ \lambda_3 = 8$
(1,2,6), (1,3,5), (2,3,4), (3,4,5), (2,4,6), (1,5,6)
This plan should be repeated four times

$v = 6,\ m = 2,\ v^* = 3,\ b = 30,\ r = 15,\ k = 3,\ n_1 = 2,\ n_2 = 1,\ n_3 = 2,\ \lambda_1 = 5,\ \lambda_2 = 0,\ \lambda_3 = 10$
(1,2,6), (1,3,5), (2,3,4), (3,4,5), (2,4,6), (1,5,6)
This plan should be repeated five times

$v = 6,\ m = 3,\ v^* = 2,\ b = 6,\ r = 4,\ k = 4,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 2,\ \lambda_2 = 2,\ \lambda_3 = 3$
(1,3,4,6), (2,3,4,5), (1,2,4,5), (1,2,3,6), (2,3,5,6), (1,4,5,6)

$v = 6,\ m = 3,\ v^* = 2,\ b = 12,\ r = 8,\ k = 4,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 4,\ \lambda_2 = 4,\ \lambda_3 = 6$
(1,3,4,6), (2,3,4,5), (1,2,4,5), (1,2,3,6), (2,3,5,6), (1,4,5,6)
This plan should be repeated twice

$v = 6,\ m = 3,\ v^* = 2,\ b = 18,\ r = 12,\ k = 4,\ n_1 = 1,\ n_2 = 2,\ n_3 = 2,\ \lambda_1 = 6,\ \lambda_2 = 6,\ \lambda_3 = 9$
(1,3,4,6), (2,3,4,5), (1,2,4,5), (1,2,3,6), (2,3,5,6), (1,4,5,6)
This plan should be repeated three times

$v = 8,\ m = 4,\ v^* = 2,\ b = 24,\ r = 9,\ k = 3,\ n_1 = 1,\ n_2 = 3,\ n_3 = 3,\ \lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 4$
(1,6,8), (2,5,7), (3,6,8), (4,5,7), (2,4,5), (1,3,6), (2,4,7), (1,3,8), (1,4,6), (2,3,5), (4,6,7), (3,5,8), (2,3,8), (1,4,7), (2,5,8), (1,6,7), (1,4,8), (2,3,7), (4,5,8), (3,6,7), (2,6,7), (1,5,8), (2,3,6), (1,4,5)

$v = 8,\ m = 2,\ v^* = 4,\ b = 6,\ r = 3,\ k = 4,\ n_1 = 3,\ n_2 = 1,\ n_3 = 3,\ \lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 2$
(1,2,7,8), (1,3,6,8), (1,4,6,7), (2,3,5,8), (2,4,5,7), (3,4,5,6)

$v = 8,\ m = 2,\ v^* = 4,\ b = 12,\ r = 6,\ k = 4,\ n_1 = 3,\ n_2 = 1,\ n_3 = 3,\ \lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = 4$
(1,2,7,8), (1,3,6,8), (1,4,6,7), (2,3,5,8), (2,4,5,7), (3,4,5,6)
This plan should be repeated twice

$v = 8,\ m = 2,\ v^* = 4,\ b = 18,\ r = 9,\ k = 4,\ n_1 = 3,\ n_2 = 1,\ n_3 = 3,\ \lambda_1 = 3,\ \lambda_2 = 0,\ \lambda_3 = 6$
(1,2,7,8), (1,3,6,8), (1,4,6,7), (2,3,5,8), (2,4,5,7), (3,4,5,6)
This plan should be repeated three times

$v = 8,\ m = 2,\ v^* = 4,\ b = 24,\ r = 12,\ k = 4,\ n_1 = 3,\ n_2 = 1,\ n_3 = 3,\ \lambda_1 = 4,\ \lambda_2 = 0,\ \lambda_3 = 8$
(1,2,7,8), (1,3,6,8), (1,4,6,7), (2,3,5,8), (2,4,5,7), (3,4,5,6)
This plan should be repeated four times

$v = 8,\ m = 2,\ v^* = 4,\ b = 30,\ r = 15,\ k = 4,\ n_1 = 3,\ n_2 = 1,\ n_3 = 3,\ \lambda_1 = 5,\ \lambda_2 = 0,\ \lambda_3 = 10$
(1,2,7,8), (1,3,6,8), (1,4,6,7), (2,3,5,8), (2,4,5,7), (3,4,5,6)
This plan should be repeated five times

$v = 8,\ m = 4,\ v^* = 2,\ b = 24,\ r = 15,\ k = 5,\ n_1 = 1,\ n_2 = 3,\ n_3 = 3,\ \lambda_1 = 6,\ \lambda_2 = 8,\ \lambda_3 = 10$
(1,3,5,6,8), (2,4,5,6,7), (1,3,6,7,8), (2,4,5,7,8), (1,2,4,5,7),(1,2,3,6,8), (2,3,4,5,7), (1,3,4,6,8), (1,3,4,6,7), (2,3,4,5,8),(1,4,5,6,7), (2,3,5,6,8), (1,2,3,5,8), (1,2,4,6,7), (2,3,5,7,8), (1,4,6,7,8), (1,4,5,7,8), (2,3,6,7,8), (1,3,4,5,8), (2,3,4,6,7), (1,2,3,6,7), (1,2,4,5,8), (2,3,5,6,7), (1,4,5,6,8)

$v = 8,\ m = 4,\ v^* = 2,\ b = 24,\ r = 12,\ k = 4,\ n_1 = 1,\ n_2 = 3,\ n_3 = 3,\ \lambda_1 = 6,\ \lambda_2 = 6,\ \lambda_3 = 4$
(1,3,7,8), (2,4,7,8), (1,3,5,6), (2,4,5,6), (3,4,5,7), (3,4,6,8), (1,2,5,7), (1,2,6,8), (1,5,6,7), (2,5,6,8), (1,3,4,7), (2,3,4,8), (3,5,7,8), (4,6,7,8), (1,2,3,5), (1,2,4,6), (1,3,4,5), (2,3,4,6), (1,5,7,8), (2,6,7,8), (2,5,6,7), (4,5,6,8), (1,2,3,7), (1,2,4,8)

$v = 8,\ m = 4,\ v^* = 2,\ b = 24,\ r = 6,\ k = 2,\ n_1 = 1,\ n_2 = 3,\ n_3 = 3,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 2$
(1,8), (2,7), (3,6), (4,5), (4,5), (3,6), (2,7), (1,8), (1,6), (2,5), (4,7), (3,8), (3,8), (4,7), (2,5), (1,6), (1,4), (2,3), (5,8), (6,9), (6,7), (5,8), (2,3), (1,4)

$v = 8,\ m = 4,\ v^* = 2,\ b = 48,\ r = 12,\ k = 2,\ n_1 = 1,\ n_2 = 3,\ n_3 = 3,\ \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 4$
(1,8), (2,7), (3,6), (4,5), (4,5), (3,6), (2,7), (1,8), (1,6), (2,5), (4,7), (3,8), (3,8), (4,7), (2,5), (1,6), (1,4), (2,3), (5,8), (6,9), (6,7), (5,8),(2,3), (1,4)
This plan should be repeated twice

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 4$, $k = 2$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 1$
(1,10), (2,9), (8,9), (7,10), (6,7), (5,8), (4,5), (3,6), (2,3), (1,4), (1,8), (2,7), (4,7), (3,8), (3,10), (4,9), (6,9), (5,10), (2,5), (1,6)

$v = 10$, $m = 5$, $v^* = 2$, $b = 40$, $r = 8$, $k = 2$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$
(1,10), (2,9), (8,9), (7,10), (6,7), (5,8), (4,5), (3,6), (2,3), (1,4), (1,8), (2,7), (4,7), (3,8), (3,10), (4,9), (6,9), (5,10), (2,5), (1,6)
This plan should be repeated twice

$v = 10$, $m = 5$, $v^* = 2$, $b = 60$, $r = 12$, $k = 2$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 3$
(1,10), (2,9), (8,9), (7,10), (6,7), (5,8), (4,5), (3,6), (2,3), (1,4), (1,8), (2,7), (4,7), (3,8), (3,10), (4,9), (6,9), (5,10), (2,5), (1,6)
This plan should be repeated three times

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 6$, $k = 3$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$
(1,3,10), (2,4,9), (1,8,9), (2,7,10), (6,7,9), (5,8,10), (4,5,7), (3,6,8), (2,3,5), (1,4,6), (1,5,8), (2,6,7), (1,4,7), (2,3,8), (3,7,10), (4,8,9), (3,6,9), (4,5,10), (2,5,9), (1,6,10)

$v = 10$, $m = 5$, $v^* = 2$, $b = 40$, $r = 12$, $k = 3$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 4$
(1,3,10), (2,4,9), (1,8,9), (2,7,10), (6,7,9), (5,8,10), (4,5,7), (3,6,8), (2,3,5), (1,4,6), (1,5,8), (2,6,7), (1,4,7), (2,3,8), (3,7,10), (4,7,8),(3,6,9), (4,5,10), (2,5,9), (1,6,10)
This plan should be repeated twice

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 8$, $k = 4$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 4$
(1,3,8,10), (2,4,7,9), (1,6,8,9), (2,5,7,10), (4,6,7,9), (3,5,8,10), (2,4,5,7), (1,3,6,8), (2,3,5,10), (1,4,6,9), (1,4,5,8), (2,3,6,7), (1,4,7,10), (2,3,8,9), (3,6,7,10), (4,5,8,9), (2,3,6,9), (1,4,5,10), (2,5,8,9), (1,6,7,10)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 8$, $k = 4$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 3$
(1,7,8,10), (2,7,8,9), (5,6,8,9), (5,6,7,10), (3,4,6,7), (3,4,5,8), (1,2,4,5), (1,2,3,6), (2,3,9,10), (1,4,9,10), (1,3,4,8), (2,3,4,7), (4,7,9,10), (3,8,9,10), (3,5,6,10), (4,5,6,9), (1,2,6,9), (1,2,5,10), (2,5,7,8), (1,6,7,8)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 10$, $k = 5$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 4$, $\lambda_2 = 4$, $\lambda_3 = 5$
(1,5,6,8,10), (2,5,6,7,9), (3,4,6,8,9), (3,4,5,7,10), (1,2,4,6,7), (1,2,3,5,8), (2,4,5,9,10), (1,3,6,9,10), (2,3,7,8,10), (1,4,7,8,9), (1,4,8,9,10), (2,3,7,9,10), (4,5,6,7,10), (3,5,6,8,9), (1,2,3,6,10), (1,2,4,5,9), (2,6,7,8,9), (1,5,7,8,10), (2,3,4,5,8), (1,3,4,6,7)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 10$, $k = 5$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = 3$
(1,3,5,9,10), (2,4,6,9,10), (1,3,7,8,9), (2,4,7,8,10), (1,5,6,7,9), (2,5,6,8,10), (3,4,5,7,9), (3,4,6,8,10), (1,2,3,5,7), (1,2,4,6,8), (1,5,7,8,9), (2,6,7,8,9), (1,3,4,5,7), (2,3,4,6,8), (1,3,7,9,10), (2,4,8,9,10), (3,5,6,7,9), (4,5,6,8,10), (1,2,3,5,9), (1,2,4,6,10)

$v = 10$, $m = 2$, $v^* = 5$, $b = 20$, $r = 10$, $k = 5$, $n_1 = 4$, $n_2 = 1$, $n_3 = 4$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 6$
(1,2,8,9,10), (1,3,7,9,10), (1,4,7,8,10), (1,5,7,8,9), (2,3,6,9,10), (2,4,6,8,10), (2,5,6,8,9), (3,4,6,7,10), (3,5,6,7,9), (4,5,6,7,8), (3,4,5,6,7), (2,4,5,6,8),
(2,3,5,6,9), (2,3,4,6,10), (1,4,5,7,8), (1,3,5,7,9), (1,3,4,7,10), (1,2,5,8,9), (1,2,4,8,10), (1,2,3,9,10)

$v = 10$, $m = 2$, $v^* = 5$, $b = 10$, $r = 5$, $k = 5$, $n_1 = 4$, $n_2 = 1$, $n_3 = 4$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 2$
(1,7,8,9,10), (2,6,8,9,10), (3,6,7,9,10), (4,6,7,8,10), (5,6,7,8,9), (2,3,4,5,6), (1,3,4,5,7), (1,2,4,5,8), (1,2,3,5,9), (1,2,3,4,10)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 12$, $k = 6$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 8$, $\lambda_2 = 6$, $\lambda_3 = 7$
(1,5,6,7,8,10), (2,5,6,7,8,9), (3,4,5,6,8,9), (3,4,5,6,7,10), (1,2,3,4,6,7), (1,2,3,4,5,8), (1,2,3,4,5,9,10), (1,2,3,6,9,10), (2,3,7,8,9,10), (1,4,7,8,9,10), (1,3,4,8,9,10), (2,3,4,7,9,10), (4,5,6,7,9,10), (3,5,6,8,9,10), (1,2,3,5,6,10), (1,2,4,5,6,9), (1,2,6,7,8,9), (1,2,5,7,8,10), (2,3,4,5,7,8), (1,3,4,6,7,8)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 12$, $k = 6$, $n_1 = 1$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = 8$
(1,3,5,6,8,10), (2,4,5,6,7,9), (1,3,4,6,8,9), (2,3,4,5,7,10), (1,2,4,6,7,9), (1,2,3,5,8,10), (2,4,5,7,9,10), (1,3,6,8,9,10), (2,3,5,7,8,10), (1,4,6,7,8,9), (1,4,5,8,9,10), (2,3,6,7,9,10), (1,4,5,6,7,10), (2,3,5,6,8,9), (1,2,3,6,7,10), (1,2,4,5,8,9), (2,3,6,7,8,9), (1,4,5,7,8,10), (2,3,4,5,8,9), (1,3,4,6,7,10)

$v = 10$, $m = 5$, $v^* = 2$, $b = 20$, $r = 14$, $k = 7$, $n_1 = 4$, $n_2 = 4$, $n_3 = 4$, $\lambda_1 = 8$, $\lambda_2 = 9$, $\lambda_3 = 10$
(1,3,5,6,7,8,9), (2,4,5,6,7,8,9), (1,3,4,5,6,8,9), (2,3,4,5,6,7,10), (1,2,3,4,6,7,9), (1,2,3,4,5,8,10), (1,2,4,5,7,9,10), (1,2,3,6,8,9,10), (2,3,5,7,8,9,10), (1,4,6,7,8,9,10), (1,3,4,5,8,9,10), (2,3,4,6,7,9,10), (1,4,5,6,7,9,10), (2,3,5,6,8,9,10), (1,2,3,5,6,7,10), (1,2,4,5,6,8,9), (1,2,3,6,7,8,9), (1,2,4,5,7,8,10), (2,3,4,5,7,8,10), (1,3,4,6,7,8,10)

$v = 12$, $m = 4$, $v^* = 3$, $b = 36$, $r = 9$, $k = 3$, $n_1 = 2$, $n_2 = 3$, $n_3 = 6$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = 2$
(1,2,12), (1,3,11), (2,3,10), (4,5,9), (4,6,8), (5,6,7), (6,7,8), (5,7,9), (4,8,9), (3,10,11), (2,10,12), (1,11,12), (1,2,9), (1,3,8), (2,3,7), (6,10,11), (5,10,12), (4,11,12), (4,5,12), (4,6,11), (5,6,10), (3,7,8), (2,7,9), (1,8,9), (1,2,6), (1,3,5), (2,3,4), (7,8,12), (7,9,11), (8,9,10), (9,10,11), (8,10,12), (7,11,12), (3,4,5), (2,4,6), (1,5,6)

$v = 12$, $m = 3$, $v^* = 4$, $b = 12$, $r = 4$, $k = 4$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 =$
(1,10,11,12), (2,9,11,12), (3,9,10,12), (4,9,10,11), (6,7,8,9), (5,7,8,10), (5,6,8,11), (5,6,7,12), (2,3,4,5), (1,3,4,6), (1,2,4,7), (1,2,3,8)

$v = 12$, $m = 3$, $v^* = 4$, $b = 24$, $r = 8$, $k = 4$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 2$
(1,10,11,12), (2,9,11,12), (3,9,10,12), (4,9,10,11), (,6,7,8,9), (5,7,8,10), (5,6,8,11), (5,6,7,12), (2,3,4,5), (1,3,4,6), (1,2,4,7), (1,2,3,8)
This plan should be repeated twice.

$v = 12$, $m = 3$, $v^* = 4$, $b = 36$, $r = 12$, $k = 4$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 6$, $\lambda_2 = 0$, $\lambda_3 = 3$
(1,10,11,12), (2,9,11,12), (3,9,10,12), (4,9,10,11), (6,7,8,9), (5,7,8,10), (5,6,8,11), (5,6,7,12), (2,3,4,5), (1,3,4,6), (1,2,4,7), (1,2,3,8)
This plan should be repeated three times.

$v = 12$, $m = 3$, $v^* = 4$, $b = 18$, $r = 6$, $k = 4$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$
(1,2,5,6), (1,3,5,7), (1,4,5,8), (2,3,6,7), (2,4,6,8), (3,4,7,8), (1,2,9,10), (1,3,9,11), (1,4,9,12), (2,3,10,11), (2,4,10,12), (3,4,11,12), (5,6,9,10), (5,7,9,11), (5,8,9,12), (6,7,10,11), (6,8,10,12), (7,8,11,12)

$v = 12$, $m = 3$, $v^* = 4$, $b = 36$, $r = 12$, $k = 4$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = 2$
(1,2,5,6), (1,3,5,7), (1,4,5,8), (2,3,6,7), (2,4,6,8), (3,4,7,8), (1,2,9,10), (1,3,9,11), (1,4,9,12), (2,3,10,11), (2,4,10,12), (3,4,11,12), (5,6,9,10), (5,7,9,11), (5,8,9,12), (6,8,9,12), (6,7,10,11), (6,8,10,12), (7,8,11,12)
This plan should be repeated twice.

$v = 12$, $m = 4$, $v^* = 3$, $b = 36$, $r = 12$, $k = 4$, $n_1 = 2$, $n_2 = 3$, $n_3 = 6$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 4$
(1,2,9,12), (1,3,8,11), (2,3,7,10), (4,5,9,12), (4,6,8,11), (5,6,7,10), (3,6,7,8), (2,5,7,9), (1,4,8,9), (3,6,10,11), (2,5,10,12), (1,4,11,12), 1,2,6,9), (1,3,5,8), (2,3,4,7), (6,9,10,11), (5,8,10,12), (4,7,11,12), (3,4,5,12), (2,4,6,11), (1,5,6,10), (3,7,8,12), (2,7,9,11), (1,8,9,10), (1,2,6,12), (1,3,5,11), (2,3,4,10), (6,7,8,12), (5,7,9,11), (4,8,9,10), (3,9,10,11), (2,8,10,12), (3,9,10,11), (2,8,10,12), (1,7,11,12), (3,4,5,9), (2,4,6,8), (1,5,6,7)

$v = 12$, $m = 4$, $v^* = 3$, $b = 36$, $r = 15$, $k = 5$, $n_1 = 2$, $n_2 = 3$, $n_3 = 6$, $\lambda_1 = 9$, $\lambda_2 = 6$, $\lambda_3 = 4$
(1,4,10,11,12), (2,5,10,11,12), (3,6,10,11,12), (1,4,7,8,9), (2,5,7,8,9), (3,6,7,8,9), (4,5,6,7,10), (4,5,6,8,11), (4,5,6,9,12),(1,2,3,7,10), (1,2,3,8,11), (1,2,3,9,12), (1,7,8,9,12), (2,7,8,9,11), (3,7,8,9,10), (1,4,5,6,10), (2,4,5,6,11), (3,4,5,6,12), (4,7,10,11,12), (5,8,10,11,12), (6,9,10,11,12), (1,2,3,4,7), (1,2,3,5,8), (1,2,3,6,9), (1,4,5,6,7), (2,4,5,6,8), (3,4,5,6,9), (1,7,10,11,12), (2,8,10,11,12), (3,9,10,11,12), (4,7,8,9,10), (5,7,8,9,11), (6,7,8,9,12), (1,2,3,4,10), (1,2,3,5,11), (1,2,3,6,12)

$v = 12$, $m = 2$, $v^* = 6$, $b = 12$, $r = 6$, $k = 6$, $n_1 = 5$, $n_2 = 1$, $n_3 = 5$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 2$
(1,8,9,10,11,12), (2,7,9,10,11,12), (3,7,8,10,11,12), (4,7,8,9,11,12), (5,7,8,9,10,12), (6,7,8,9,10,11), (2,3,4,5,6,7), (1,3,4,5,6,8), (1,2,4,5,6,9), (1,2,3,5,6,10), (1,2,3,4,6,11), (1,2,3,4,5,12)

$v = 12$, $m = 2$, $v^* = 6$, $b = 24$, $r = 12$, $k = 6$, $n_1 = 5$, $n_2 = 1$, $n_3 = 5$, $\lambda_1 = 8$, $\lambda_2 = 0$, $\lambda_3 = 4$
(1,8,9,10,11,12), (2,7,9,10,11,12), (3,7,8,10,11,12), (4,7,8,9,11,12), (5,7,8,9,10,12), (6,7,8,9,10,11), (2,3,4,5,6,7), (1,3,4,5,6,8), (1,2,4,5,6,9), (1,2,3,5,6,10), (1,2,3,4,6,11), (1,2,3,4,5,12)
This plan should be repeated twice.

$v = 12$, $m = 2$, $v^* = 6$, $b = 20$, $r = 10$, $k = 6$, $n_1 = 5$, $n_2 = 1$, $n_3 = 5$, $\lambda_1 = 4$, $\lambda_2 = 0$, $\lambda_3 = 6$
(1,2,5,9,10,12), (1,2,6,9,10,11), (1,2,6,9,10,11), (1,3,4,8,11,12), (1,3,6,8,10,11), (1,4,5,8,9,12), (2,3,4,7,11,12), (2,3,5,7,10,12), (2,4,6,7,9,11), (3,5,6,7,8,10), (4,5,6,7,8,9), (3,4,6,7,8,11), (3,4,5,7,8,12), (2,5,6,7,9,10), (2,4,5,7,9,12), (2,3,6,7,10,11), (1,5,6,8,9,10), (1,4,6,8,9,11), (1,3,5,8,10,12), (1,2,4,9,11,12), (1,2,3,10,11,12)

$v = 12$, $m = 2$, $v^* = 6$, $b = 30$, $r = 15$, $k = 6$, $n_1 = 5$, $n_2 = 1$, $n_3 = 5$, $\lambda_1 = 7$, $\lambda_2 = 0$, $\lambda_3 = 8$
(1,2,9,10,11,12), (1,3,8,10,11,12), (1,4,8,9,11,12), (1,5,8,9,10,12), (1,6,8,9,10,11), (2,3,7,10,11,12), (2,4,7,9,11,12), (2,5,7,9,10,12), (2,6,7,9,10,11), (3,4,7,8,11,12), (3,5,7,8,10,12), (3,6,7,8,10,11), (4,5,7,8,9,12), (4,6,7,8,9,11), (5,6,7,8,9,10), (3,4,5,6,7,8), (2,4,5,6,7,9), (2,3,5,6,7,10), (2,3,4,6,7,11), (2,3,4,5,7,12), (1,4,5,6,8,9), (1,3,5,6,8,10), (1,3,4,6,8,11), (1,3,4,5,8,12), (1,2,5,6,9,10), (1,2,4,6,9,11), (1,2,4,5,9,12), (1,2,3,6,10,11), (1,2,3,5,10,12), (1,2,3,4,11,12)

$v = 12$, $m = 3$, $v^* = 4$, $b = 12$, $r = 6$, $k = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 4$, $\lambda_2 = 3$, $\lambda_3 = 2$
(1,5,9,10,11,12), (2,6,9,10,11,12), (3,7,9,10,11,12), (4,8,9,10,11,12), (1,5,6,7,8,9), (2,5,6,7,8,10), (3,5,6,7,8,11), (4,5,6,7,8,12), (1,2,3,4,5,9), (1,2,3,4,6,10), (1,2,3,4,7.11), (1,2,3,4,8,12)

$v = 12$, $m = 3$, $v^* = 4$, $b = 24$, $r = 12$, $k = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 8$, $\lambda_2 = 6$, $\lambda_3 = 4$
(1,5,9,10,11,12), (2,6,9,10,11,12), (3,7,9,10,11,12), (4,8,9,10,11,12), (1,5,6,7,8,9), (2,5,6,7,8,10), (3,5,6,7,8,11), (4,5,6,7,8,12), (1,2,3,4,5,9), (1,2,3,4,6,10), (1,2,3,4,7.11), (1,2,3,4,8,12)
This plan should be repeated twice.

$v = 12$, $m = 3$, $v^* = 4$, $b = 18$, $r = 9$, $k = 6$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 5$
(1,2,5,6,11,12), (1,3,5,7,10,12), (1,4,5,8,10,11), (2,3,6,7,9,12), (2,4,6,8,9,11), (3,4,7,8,9,10), (1,2,7,8,9,10), (1,3,6,8,9,10), (1,4,6,7,9,12), (2,3,5,8,10,11), (2,4,5,7,10,12), (3,4,5,6,11,12), (3,4,5,6,9,10), (2,4,5,7,9,11), (2,3,5,8,9,12), (1,4,6,7,10,11), (1,3,6,8,10,12), (1,2,7,8,11,12)

$v = 12$, $m = 6$, $v^* = 2$, $b = 30$, $r = 15$, $k = 6$, $n_1 = 1$, $n_2 = 5$, $n_3 = 5$, $\lambda_1 = 10$, $\lambda_2 = 7$, $\lambda_3 = 6$
(1,3,9,10,11,12), (2,4,9,10,11,12), (1,5,7,8,11,12), (2,6,7,8,11,12), (1,7,8,9,10,12), (2,7,8,9,10,12), (1,5,6,9,11,12), (2,5,6,9,11,12), (1,5,6,7,9,10), (2,5,6,8,9,10), (5,6,7,8,9,11), (5,6,7,8,10,12), (3,4,5,9,11,12), (3,4,6,10,11,12), (3,4,7,9,10,11), (3,4,8,9,10,12), (3,4,5,7,8,11), (3,4,6,7,8,12), (3,4,5,6,7,9), (3,4,5,6,8,10), (1,2,3,5,11,12), (1,2,4,6,11,12), (1,2,3,7,9,10), (1,2,4,8,9,10),(1,2,3,7,8,11), (1,2,4,7,8,12), (1,2,3,5,6,9), (1,2,4,5,6,10), (1,2,3,4,5,7), (1,2,3,4,6,8)

$v = 12$, $m = 3$, $v^* = 4$, $b = 12$, $r = 8$, $k = 8$. $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 6$, $\lambda_2 = 4$, $\lambda_3 = 5$
(1,5,6,7,8,10,11,12), (2,5,6,7,8,9,11,12), (3,5,6,7,8,9,10,12), (4,5,6,7,8,9,10,11), (1,2,3,4,6,7,8,9), (1,2,3,4,5,7,8,10), (1,2,3,4,5,6,8,11), (1,2,3,4,5,6,7,12), (2,3,4,5,9,10,11,12), (1,3,4,6,9,10,11,12), (1,2,4,7,9,10,11,12), (1,2,3,8,9,10,11,12)

$v = 12$, $m = 3$, $v^* = 4$, $b = 18$, $r = 12$, $k = 8$, $n_1 = 3$, $n_2 = 2$, $n_3 = 6$, $\lambda_1 = 8$, $\lambda_2 = 9$, $\lambda_3 = 7$
(1,2,5,6,9,10,11,12), (1,3,5,7,9,10,11,12), (1,4,5,8,9,10,11,12), (2,3,6,7,9,10,11,12), (2,4,6,8,9,10,11,12), (3,4,7,8,9,10,11,12), (1,2,5,6,7,8,9,10), (1,3,5,6,7,8,9,11), (1,4,5,6,7,8,9,12), (2,3,5,6,7,8,10,11), (2,4,5,6,7,8,10,11,12), (3,4,5,6,7,8,11,12), (1,2,3,4,5,6,9,11), (1,2,3,4,5,7,9,11), (1,2,3,4,5,8,9,12), (1,2,3,4,6,7,10,11), (1,2,3,4,6,8,10,12), (1,2,3,4,7,8,11,12).

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Konstrukcja układów blokowych typu prostokątnego przydatnych w doświadczeniach dwuczynnikowych

STRESZCZENIE

Praca dotyczy planowania doświadczeń w częściowo zrównoważonych układach blokowych niekompletnych o prostokątnym schemacie partnerstwa z trzema klasami partnerów (RT PBIB). Podano pewne metody konstrukcji układów RT PBIB, które mogą znaleźć zastosowania w dowolnych doświadczeniach dwuczynnikowych. Macierze incydencji tych układów mogą być przedstawione jako rozszerzenie binarnego iloczynu Kroneckera odpowiednio dobranych macierzy. Zwrócono uwagę na statystyczne własności tych układów, związane z estymacją bazowych i elementarnych kontrastów obiektowych. Zamieszczono katalog układów RT PBIB dla 6, 8, 10 i 12 obiektów, których liczba replikacji r spełnia warunek $2 \leq r \leq 15$.

SŁOWA KLUCZOWE: doświadczenia dwuczynnikowe, iloczyn Kroneckera macierzy, macierze zrównoważenia, układy RT PBIB